Calculus 2

13 March 2014, 14:00-16:00

university of groningen

Midterm Exam

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**5+10 Points.**]

   Consider the function
   $$f(x,y) = \frac{x-y}{x^2-y^2}$$
   with domain $D = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 \neq y^2\}$.

   (a) Determine the limit of f at $(x,y) = (1,1)$.

   (b) Can the function be extended to the domain $\mathbb{R}^2$ in such a way that it is continuous at every point $(x,y) \in \mathbb{R}^2$? Justify your answer.

2. [**10+5+10 Points.**]

   Consider the curve parametrized by $\mathbf{r} : [0,1] \to \mathbb{R}^3$ with
   $$\mathbf{r}(t) = t\,\mathbf{i} + \frac{\sqrt{2}}{2}t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}.$$

   (a) Determine the length of the curve.

   (b) For each point on the curve, determine a unit tangent vector.

   (c) At each point on the curve, determine the curvature of the curve.

3. [**10+15 Points.**]

   Consider the ellipsoid
   $$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

   (a) Determine the tangent plane of the ellipsoid at the point $(x,y,z) = (-2,-1,3)$.

   (b) Use the method of Lagrange multipliers to find the points on the ellipsoid closest to and farthest away from the origin.

4. [**25 Points.**]

   Determine the volume of the region enclosed by the paraboloids $z = 9 - x^2 - y^2$ and $z = 3x^2 + 3y^2 - 16$. To this end note that the paraboloids intersect in a circle contained in a plane of constant z. A sketch might help.

1. (a) Let $(x,y) \in D$.

Then $f(x,y) = \dfrac{x-y}{x^2-y^2} = \dfrac{x-y}{(x-y)(x+y)} = \dfrac{1}{x+y}$

$$\lim_{(x,y)\to(1,1)} \frac{1}{x+y} = \frac{1}{2}$$

(b) The function cannot be extended to $\mathbb{R}^2$, because f has no limit at all $(x,y) \in \mathbb{R}^2$ where $y = -x$. Consider, e.g., the line $(x(t), y(t)) = (t + x_0, t - x_0)$ which for $t = 0$ intersects the line $y = -x$ perpendicularly at $(x,y) = (x_0, -x_0)$.

Then $f(x(t), y(t)) = \dfrac{1}{x(t)+y(t)} = \dfrac{1}{2t}$

which has no limit at $t = 0$.

2. (a) $r'(t) = i + \sqrt{2}\,t\,j + t^2 k$

$$L = \int_0^1 |r'(t)|\,dt = \int_0^1 (1 + 2t^2 + t^4)^{1/2}\,dt$$

$$= \int_0^1 ((1+t^2)^2)^{1/2}\,dt = \int_0^1 (1+t^2)\,dt$$

$$= \left(t + \frac{1}{3}t^3\right)\Big|_{t=0}^{t=1} = 1\frac{1}{3} = \frac{4}{3}$$

(b) unit tangent vector at $r(t)$:

$$T = \frac{r'(t)}{|r'(t)|} = \frac{1}{1+t^2}\left(i + \sqrt{2}\,t\,j + t^2 k\right)$$

(c) $$\kappa = \left|\frac{dT}{ds}\right| = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{1+t^2} \cdot \left|\frac{d}{dt}\left(\frac{1}{1+t^2} i + \frac{\sqrt{2}\,t}{1+t^2} j + \frac{t^2}{1+t^2} k\right)\right|$$

$$= \frac{1}{1+t^2}\left| -\frac{2t}{(1+t^2)^2} i + \sqrt{2}\,\frac{1-t^2}{(1+t^2)^2} j + \frac{2t}{(1+t^2)^2} k \right|$$

$$= \frac{1}{1+t^2} \cdot \frac{\left(2\cdot(1+t^2)^2\right)^{1/2}}{(1+t^2)^2} = \frac{\sqrt{2}}{(1+t^2)^2}$$

3. (a) First method: the ellipsoid is a level set of the function

$$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

F has gradient

$$\nabla F(x,y,z) = \left(\frac{x}{2}, 2y, \frac{2}{9}z\right)$$

at $(x,y,z) = (-2,-1,3)$:

$$\nabla F(-2,-1,3) = \left(-1, -2, \frac{2}{3}\right)$$

The tangent plane is then given by

$$\left(-1,-2,\frac{2}{3}\right)\cdot\left(x-(-2), y-(-1), 3-z\right) = 0$$

$$\Leftrightarrow \quad -x-2 -2y-2 + 2 - \frac{2}{3}z = 0$$

$$\Leftrightarrow \quad x + 2y + \frac{2}{3}z = -2$$

Second method: the upper half of the ellipsoid is the graph of the function $f(x,y) = 3\sqrt{3 - \frac{x^2}{4} - y^2}$

The tangent plane is then obtained from the linearization of f at $(-2,-1)$:

$$(*) \quad z = f(-2,-1) + f_x(-2,-1)(x-(-2)) + f_y(-2,-1)(y-(-1))$$

we have: $f(-2,-1) = 3$

$$f_x(x,y) = -\frac{3}{4}\frac{x}{\sqrt{3-\frac{x^2}{4}-y^2}} \Rightarrow f_x(-2,-1) = \frac{3}{2}$$

$$f_y(x,y) = -3\frac{y}{\sqrt{3-\frac{x^2}{4}-y^2}} \Rightarrow f_y(-2,1) = 3$$

$\Rightarrow (*)$ becomes: $z = 3 + \frac{3}{2}(x+2) + 3(y+1)$ which agrees with above

3. (b) Let $g(x,y,z) = x^2+y^2+z^2$ (square distance to the origin)

Method of Lagrange multipliers:

$$\left.\begin{array}{l} F_x = \lambda g_x \\ F_y = \lambda g_y \\ F_z = \lambda g_z \\ F(x,y,z) = 3 \end{array}\right\} \iff \left.\begin{array}{l} \frac{x}{2} = \lambda 2x \\ 2y = \lambda 2y \\ \frac{2}{9} z = \lambda 2z \\ \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \end{array}\right\}$$

$$\iff \left.\begin{array}{l} x = 0 \text{ or } \lambda = \frac{1}{4} \\ y = 0 \text{ or } \lambda = 1 \\ z = 0 \text{ or } \lambda = \frac{1}{9} \\ \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \end{array}\right\}$$

Only possibilities:

$x = y = 0$ which implies $z = \pm\sqrt{27}$ (by last equation)

$x = z = 0$ — " — $y = \pm\sqrt{3}$ (— " —)

$y = z = 0$ — " — $x = \pm\sqrt{12}$ (— " —)

Fill in these points in g:

$g(0,0,\pm\sqrt{27}) = 27$ max.

$g(0,\pm\sqrt{3},0) = 3$ min.

$g(\pm\sqrt{12},0,0) = 12$

So the points of maximal distance are $(x,y,z) = (0,0,\pm\sqrt{27})$

— " — minimal — " — $(x,y,z) = (0,\pm\sqrt{3},0)$

4. Determine circle of intersection:

$$9 - x^2 - y^2 = 3x^2 + 3y^2 - 16$$

$$\Leftrightarrow \quad 25 = 4(x^2 + y^2)$$

circle of radius $\frac{5}{2}$

Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq \left(\frac{5}{2}\right)^2\}$

The volume of the region enclosed by the paraboloids then is

$$\iint_D \int_{3x^2+3y^2-16}^{9-x^2-y^2} dt \, dx \, dy = \iint_D \left[9 - x^2 - y^2 - (3x^2 + 3y^2 - 16)\right] dx \, dy$$

$$= \iint_D \left[25 - 4x^2 - 4y^2\right] dx \, dy$$

in polar coordinates

$$= \int_0^{5/2} \int_0^{2\pi} (25 - 4r^2) \, r \, d\theta \, dr$$

$$= 2\pi \left(25 \frac{r^2}{2} - r^4\right)\Big|_0^{5/2}$$

$$= 2\pi \left(\frac{25^2}{8} - \frac{25^2}{16}\right)$$

$$= \pi \frac{625}{8}$$